

Bell's inequality: Physics meets Probability

Andrei Khrennikov

International Center for Mathematical Modeling
in Physics, Engineering and Cognitive Science,
Växjö University, S-35195, Sweden
email: Andrei.Khrennikov@vxu.se

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Abstract

In this review we remind the viewpoint that violation of Bell's inequality might be interpreted not only as an evidence of the alternative – either nonlocality or “death of reality” (under the assumption the quantum mechanics is incomplete). Violation of Bell's type inequalities is a well known sufficient condition of probabilistic incompatibility of random variables – impossibility to realize them on a single probability space. Thus, in fact, we should take into account an additional interpretation of violation of Bell's inequality – a few pairs of random variables (two dimensional vector variables) involved in the EPR-Bohm experiment are incompatible. They could not be realized on a single Kolmogorov probability space. Thus one can choose between: a) completeness of quantum mechanics; b) nonlocality; c) “death of reality”; d) non-Kolmogorovness. In any event, violation of Bell's inequality has a variety of possible interpretations. Hence, it could not be used to obtain the definite conclusion on the relation between quantum and classical models.

Keywords: EPR experiment, EPR-Bohm experiment, Bell's inequality, probability measure, probabilistic compatibility and incompatibility, Kolmogorov probability space, contextuality, detectors efficiency, fair and unfair sampling, negative probabilities, rejection of the photon hypothesis, frequency probabilities

1 Introduction

This paper was stimulated by the recent review of Genovese [1] devoted to EPR and EPR-Bohm experiments, Bell's inequality, quantum nonlocality, realism and all those questions which are nowadays intensively discussed, see [2]–[17] for foundations and e.g. [18]–[23] for recent debates. Genovese presented an interesting and deep analysis of these fundamental problems. However, Genovese's presentation and conclusions were standard for a typical physical presentation of the Bell's arguments.

The aim of this paper is to remind to the physical community (especially, its quantum information part) that Bell's inequality is an important point where *probability theory meets physics*. Unfortunately, the fundamental role of probability in this framework is missed. In any event physicists try to escape coupling of mentioned fundamental physical problems with foundations of probability theory.

The situation in quantum physics is such as it could be in general relativity if the modern mathematical formalization of geometry were ignored. For example, assume that one would not know that geometry is not reduced to the Euclidean geometry (that there exists e.g. the Lobachevsky geometry). In such a case by finding “non-Euclidean behavior” she might make the conclusion about “death of reality”. In some way it would really be “death of reality”, but only Euclidean reality. Non-Euclidean local effects might be also imagine as nonlocal Euclidean effects. In probability theory the Kolmogorov probability model is an analogue of the Euclidean geometry. In this review we present the viewpoint that violation of Kolmogorovness might be interpreted as “death or reality” or nonlocality. However, this is death of only Kolmogorovian reality. Kolmogorov nonlocality might be in fact simply non-Kolmogorov locality.

We present results of studies on the problem of *probabilistic compatibility of a family of random variables*. They were done during last hundred years. And they have the direct relation to Bell's inequality. However, a priory studies on probabilistic compatibility have no direct relation to the well known fundamental problems which are typically discussed by physicists, namely, *realism and locality* [2]–[17], see e.g. [18]–[23] for recent debates. After a review on conditions of probabilistic compatibility, we shall try to find traces of this probabilistic research in physics, e,g,. parameters of measurement devices, negative probabilities, detectors efficiency, fair sampling, rejection of photon (Lamb's anti-photon, Santos' views).

We remark that our considerations would not imply that the conventional interpretation of Bell's inequality [2]–[17] should be rejected. In principle, Bell's conditions (nonlocality, “death of reality”) could also be taken into account. Our aim is to show that Bell's conditions are only *sufficient, but not necessary* for violation of Bell's inequality.

Therefore other interpretations of violation of this inequality are also possible. Bell's alternative – either quantum mechanics or local realism – can be extended – either existence of a single probability measure¹ for incompatible experimental contexts or quantum mechanics. We notice that existence of such a single probability was never assumed in classical (Kolmogorov) probability theory, but it was used by J. Bell to derive his inequality (it was denoted by ρ in Bell's derivation). Therefore if one wants to use Bell's inequality, he should find reasonable arguments supporting Bell's derivation, roughly speaking:

Why do we use such an assumption in quantum physics, although we have never used it in classical probability theory?

In classical probability theory researchers never try to put statistical data collected on the basis of different sampling experiments into one single probability space. However, in quantum mechanics we (at least Bell and his adherents) try to do this. From the point of view of “the probabilistic opposition” to the conventional interpretation of violation of Bell's inequality, the crucial problem of Bell's considerations was placing statistical data collected in a few totally different experiments (corresponding to different setting of polarization beam splitters) in one probabilistic inequality. We remark that the same trick was done by Feynman [24] in his probabilistic analysis of the two slit experiment. However, Feynman remarked that his analysis demonstrated violation of laws of classical probability theory.² And this is not surprising, because he proceeded against all rules of the conventional probabilistic practice. Kolmogorov emphasized from the very beginning [27] that any experimental context induces its own probability space. But Feynman considered three different contexts: C_{12} two slits are open, C_1 only the first slit is open, C_2 only the second slit is open. By some reason (I think sim-

¹By using the terminology of modern probability theory one should speak about existence of a single Kolmogorov probability space [27], [28].

²He had no idea about the rigorous axiomatics of modern classical probability theory (the Kolmogorov model). Therefore he wrote about violation of laws of classical Laplacian probability.

ply because of lack of education in probability) he wanted to put such data collected under three different contexts into a single probability space. The impossibility to do this Feynman interpreted as astonishing violation of laws of classical probability theory. To solve this problem, he decided to assume (in accordance with fathers of quantum mechanics, see e.g. Dirac [25]) that each quantum particle interferes with itself.³ Bell did more or less the same thing with the EPR-Bohm experiment. In contrast to Feynman, Bell did not even see the probabilistic inconsistency of his considerations. This story was presented in detail in my book [26] (which is definitely unreadable for physicists, because of too much probability).

This paper is based on the results of research on the probabilistic structure of Bell's inequality. We can call this (very inhomogeneous) group of researchers "probabilistic opposition"⁴: Accardi [29]–[31], Fine [32], Garola and Solombrino [33]–[35], Hess and Philipp [36], Khrennikov [26], [37]–[40], Kupczynski [41]–[43], Pitowsky [44], [45], Rastal [46], Sozzo [66]. On one hand, it is amazing that so many people came to the same conclusion practically independently. On the other hand, it is also amazing that this conclusion is not so much known by physicists (even for mathematically interested researchers working in quantum information theory). There is definitely a problem of communication. I hope that this review would inform physicists about some general mathematical ideas on Bell's inequality.

I remark that (to my knowledge) only one physicist (and moreover very good experimenter!), Klyshko [47]–[47], obtained similar conclusion – independently from mathematicians! He did not know anything about mentioned research in probabilistic analysis of Bell's arguments. However, he

³Such a conclusion is in visible contradiction with his own path integral approach to quantum mechanics. Of course, one might say (as Feynman did) that these trajectories are virtual. However, in such a case one should explain such a funny coincidence of prediction of the quantum operator formalism with the result of integration with respect to these totally nonphysical trajectories. Feynman did not do this. It is clear that we do not know all historical details. It might be that at the beginning Feynman assigned to these trajectories more physical meaning. We only know that his path integral approach was terribly criticized by Bohr. Feynman was able to sell his path integral approach to Bohr (and hence to the quantum community) only through personal contacts with Pauli. It is clear that the conflict with Bohr was not possible to resolve in the framework of physical trajectories.

⁴Names are presented in the alphabetic order. In spite the evident fact that physicists totally ignore this probabilistic research, the "probabilistic opposition" is permanently disturbed by conflicts on priority.

also pointed out to non-Kolmogorovness as a reasonable alternative to non-locality, “death of reality” and completeness of quantum mechanics.

I also point out to a series of papers of Andreev and Manko et al. [51]–[55] who used quantum tomographic [56]–[60] interpretation Bell’s inequality. Although they have never emphasized “non-Kolmogorovness” of their approach, it is evident that the quantum tomographic scheme for Bell’s inequality is based on a family of probability measures associated with involved experimental contexts. It is impossible to construct a single probability serving for the total collection of statistical data.

Finally, we note that Volovich [61], [62] pointed out to the role of space variables in the EPR-Bohm studies (which was surprisingly missed from modern considerations). He also emphasized the crucial difference between the original EPR experiment for correlations of positions and momenta and its Bohm version for photon polarization or electron spin. Khrennikov and Volovich demonstrated in the rigorous probabilistic framework that the original EPR experiment for measurement of continuous variables for entangled systems (position and momentum) differs crucially from its EPR-Bohm version for measurement of discrete variables for entangled systems (projections of polarization or spin). We discuss this point in the present review.

Since in this paper we shall discuss Bell’s proof of its inequality and its versions, we present (for reader’s convenience) these proofs in the appendix. We remark that the analysis of probabilistic assumptions of Bell’s arguments is extremely important for modern quantum physics, especially quantum information, cryptography and computing. Consequences of the modern interpretation of violation of Bell’s inequality for foundations of quantum mechanics (and nanotechnology) are really tremendous. Hence, conditions of derivation of this inequality should be carefully checked. In this paper considerations are concentrated on the analysis of the possibility to use a single probability distribution underlying two dimensional marginal distributions predicted theoretically by quantum mechanics.

We note that the present experimental situation for the EPR-Bohm experiment is very complicated. It is well known that Aspect et al. [8], see also Weihs et al. [16], [17], shown that Bell’s inequality is really violated by the experimental data collected in four experiments corresponding to choices of different settings of polarization. However, recently Adenier and Khrennikov found that it was not the end of this great experimental story. By analyzing the data from the first experiment that closed locality loophole, see Weihs et al. [16], we found that data contains anomalies of the following type. The

Bell's expression for correlations is a linear combination of two dimensional probability distributions for polarization. The astonishing fact is that these experimental two dimensional probability distributions essentially differ from those which are predicted by quantum mechanics theoretically, see [93], [94]. In some mysterious way these anomalous deviations are cancelled (by compensating each other) in the Bell's expression for correlations. It is even more astonishing that the same anomalies were present in statistical data from the pioneer experiment of Aspect et al. [8] which differs crucially by its technical realization from the experiment in Weihs et al. [16]. This fact about statistical anomalies was not communicated in the article [8]. However, it can be found in the PhD-thesis of Alain Aspect, see [9].

Therefore new cleaner experiments should be done in future. The common opinion that Bell's arguments are totally confirmed by experiments (with just such a purely technological problem as inefficiency of detectors) is far from experimental reality.

Recently Bell-type inequalities for tests of compatibility of *nonlocal realistic models* with quantum mechanics were derived, see Leggett [95]. They were generalized and tested experimentally by Gröblacher et al. [96]. It was an important step toward unification of positions of “orthodox quantum community” and mentioned above “probabilistic opposition.” Physicists started to understand that Bell's condition of locality played a subsidiary role in business with Bell-type inequalities. Even without the locality condition one can obtain Bell-type inequalities. The crucial condition (as it was pointed out by e.g. Fine [32] and Rastal [46], see [26] for detailed presentation) is the existence of a single probability measure serving all experimental settings involved in a Bell-type inequality – probabilistic compatibility of random variables. Unfortunately, the latter point of view was again missed by Leggett [95] and Gröblacher et al. [96], see section 14.

2 Studies of Boole and Vorobjev on probabilistic compatibility

Consider a system of three random variables $a_i, i = 1, 2, 3$. Suppose for simplicity that they take discrete values and moreover they are dichotomous: $a_i = \pm 1$. Suppose that these variables as well as their pairs can be measured

and hence joint probabilities for pairs are well defined:

$$P_{a_i, a_j}(\alpha_i, \alpha_j) \geq 0$$

and

$$\sum_{\alpha_i, \alpha_j = \pm 1} P_{a_i, a_j}(\alpha_i, \alpha_j) = 1.$$

Question: *Is it possible to construct the joint probability distribution, $P_{a_1, a_2, a_3}(\alpha_1, \alpha_2, \alpha_3)$, for any triple of random variables?*

Surprisingly this question was asked and answered for hundred years ago by Boole (who invented Boolean algebras). This was found by Itamar Pitowsky [64], [65], see also preface [20]. To study this problem, Boole derived inequality which coincides with the well known in physics Bell's inequality. Violation of this Boole-Bell inequality implies that for such a system of three random variables the joint probability distribution $P_{a_1, a_2, a_3}(\alpha_1, \alpha_2, \alpha_3)$ does not exist.

Example. (see [67]) Suppose that

$$P(a_1 = +1, a_2 = +1) = P(a_1 = -1, a_2 = -1) = 1/2;$$

$$P(a_1 = +1, a_3 = +1) = P(a_1 = -1, a_3 = -1) = 1/2;$$

$$P(a_2 = +1, a_3 = -1) = P(a_2 = -1, a_3 = +1) = 1/2.$$

Hence, $P(a_1 = +1, a_2 = -1) = P(a_1 = -1, a_2 = +1) = 0$; $P(a_1 = +1, a_3 = -1) = P(a_1 = -1, a_3 = +1) = 0$, $P(a_2 = +1, a_3 = +1) = P(a_2 = -1, a_3 = -1) = 0$. Then it is impossible to construct a probability measure which would produce these marginal distributions. We can show this directly [67]. Suppose that one can find a family of real constants $P(\epsilon_1, \epsilon_2, \epsilon_3)$, $\epsilon_j = \pm 1$, such that

$$P(\epsilon_1, \epsilon_2, +1) + P(\epsilon_1, \epsilon_2, -1) = P(a_1 = \epsilon_1, a_2 = \epsilon_2), \dots,$$

$$P(+1, \epsilon_2, \epsilon_3) + P(-1, \epsilon_2, \epsilon_3) = P(a_2 = \epsilon_2, a_3 = \epsilon_3).$$

Then he immediately finds that some of these numbers should be negative. In a more fashionable way one can apply Bell's inequality for correlations, see appendix: $|\langle a_1, a_2 \rangle - \langle a_2, a_3 \rangle| \leq 1 - \langle a_1, a_3 \rangle$. We have:

$$\langle a_1, a_2 \rangle = 1; \langle a_1, a_3 \rangle = 1; \langle a_2, a_3 \rangle = -1.$$

Bell's inequality should imply: $1 - (-1) = 2 \leq 1 - 1 = 0$. We remark that in accordance with Boole we consider Bell's inequality just as a necessary condition for probabilistic compatibility.

Thus Bell's inequality was known in probability theory. It was derived as a constraint which violation implies nonexistence of the joint probability distribution.

Different generalizations of this problem were studied in probability theory. The final solution (for a system of n random variable) was obtained by Soviet mathematician Vorobjev [67] (as was found by Hess and Philipp [36]). His result was applied in purely macroscopic situations – in game theory and optimization theory.

We emphasize that for mathematicians consideration of Bell's type inequalities did not induce revolutionary reconsideration of laws of nature. The joint probability distribution does not exist just because those observables could not be measured simultaneously.

3 The EPR-Bohm Experiment, Impossibility to Measure Three Polarization Projections Simultaneously

We consider now one special application of Boole's theorem the EPR-Bohm experiment for measurements of spin projections for pairs of entangled photons.⁵ Denote corresponding random variables by a_θ^1 and a_θ^2 , respectively (the upper index $k = 1, 2$ denotes observables for corresponding particles in a pair of entangled photons). Here θ is the angle parameter determining the setting of polarization beam splitter. For our purpose it is sufficient to consider three different angles: $\theta_1, \theta_2, \theta_3$. (In fact, for real experimental tests we should consider four angles, but it does not change anything in our considerations).

By using the condition of precise correlation for the singlet state we can

⁵Although both Boole's and Bell's theorems are based on the same inequality, the conclusions are totally different. These are “nonexistence of the joint probability distribution” and “either local realism or quantum mechanics”, respectively. Thus we would like to analyze the EPR-Bohm experiment from the viewpoint of Boole (Vorobjev, Accardi, Fine, Pitowsky, Rastal, Hess and Philipp and the author).

identify observables

$$a_\theta(\lambda) \equiv a_\theta^1(\lambda) = a_\theta^2(\lambda).$$

The following discrete probability distributions are well defined: $P_{a_\theta}(\alpha)$ and $P_{a_{\theta_i}, a_{\theta_j}}(\alpha, \beta)$. Here $\alpha, \beta = \pm 1$. We remark that in standard derivations of Bell's type inequality for probabilities (and not correlations), see appendix, there are typically used the following symbolic expressions of probabilities: $P(a_\theta(\lambda) = \alpha)$ and $P(a_{\theta_i}(\lambda) = \alpha, a_{\theta_j}(\lambda) = \beta)$. However, by starting with a single probability P (defined on a single space of “hidden variables” Λ) we repeat Bell's schema (which we would not like to repeat in this paper).

Thus we are precisely in the situation which was considered in probability theory. Boole (and Vorobjev) would ask: Do polarization-projections for any triple of angles have the joint probability distribution? Can one use a single probability measure P ? The answer is negative – because the Boole-Bell inequality is violated (or because necessary condition of Vorobjev theorem is violated). Thus it is impossible to introduce the joint probability distribution for an arbitrary triple of angles.

On the other hand, Bell started his considerations with the assumption that such a single probability measure exists, see appendix. He represented all correlations as integrals with respect to the same probability measure ρ :

$$\langle a_{\theta_i}, a_{\theta_j} \rangle = \int_{\Lambda} a_{\theta_i}(\lambda) a_{\theta_j}(\lambda) dP(\lambda).$$

(We shall use the symbol P , instead of Bell's ρ to denote probability).

In opposite to Bell, Boole would not be so much excited by evidence of violation of Bell's inequality in the EPR-Bohm experiment. The situation when pairwise probability distributions exist, but a single probability measure P could not be constructed is rather standard. What would be a reason for existence of P in the case when the simultaneous measurement of three projections of polarization is impossible?

A priori nonexistence of P has nothing to do with nonlocality or “death or reality.” The main problem is not the assumption that polarization projections are represented in the “local form”:

$$a_{\theta_i}^1(\lambda), a_{\theta_j}^2(\lambda)$$

and not in the “nonlocal form”

$$a_{\theta_i}^1(\lambda | a_{\theta_j}^2 = \beta), a_{\theta_j}^2(\lambda | a_{\theta_i}^1 = \alpha),$$

where $\alpha, \beta = \pm 1$. The problem is nor assigning to each λ the definite value of the random variable – “realism.”

The problem is impossibility to realize three random variables

$$a_{\theta_1}(\lambda), a_{\theta_2}(\lambda), a_{\theta_3}(\lambda)$$

on the same space of parameters Λ with same probability measure P . By using the modern terminology we say that it is impossible to construct a Kolmogorov probability space for such three random variables.

In this situation it would be reasonable to find sources of nonexistence of a Kolmogorov probability space. We remark that up to now we work in purely classical framework– neither the ψ -function or noncommutative operators were considered. We have just seen [8], [16], [17] that experimental statistical data violates the necessary condition for the existence of a single probability P . Therefore it would be useful to try to proceed purely classically in the probabilistic analysis of the EPR-Bohm experiment. We shall do this in the next section.

4 Contextual Analysis of the EPR-Bohm Experiment

As was already emphasized in my book [26], the crucial point is that in this experiment one combine statistical data collected on the basis of three different complexes of physical conditions (contexts). We consider context C_1 – setting θ_1, θ_2 , context C_2 – setting θ_1, θ_3 , and finally context C_3 – setting θ_2, θ_3 . We recall that already in Kolmogorov’s book [27] (where the modern axiomatics of probability theory was presented) it was pointed out that each experimental context determines its own probability space. By Kolmogorov in general three contexts $C_j, j = 1, 2, 3$, should generate three Kolmogorov spaces: with sets of parameters Ω_j and probabilities P_j .

The most natural way to see the source of appearance of such spaces is to pay attention to the fact that (as it was underlined by Bohr) the result of measurement is determined not only by the initial state of a system (before measurement), but also by the *whole measurement arrangement*. Thus states of measurement devices are definitely involved. We should introduce not only space Λ of states of a system (a pair of photons), but also spaces of states of polarization beam splitters – Λ_θ . (We proceed under the assumption that

the state of polarization beam splitter depends only on the orientation θ . In principle, we should consider two spaces for each θ for the first and the second splitters. In reality they are not identical.) Thus, see [26], for the context C_1 the space of parameters (“hidden variables”) is given by

$$\Lambda_1 = \Lambda \times \Lambda_{\theta_1} \times \Lambda_{\theta_2},$$

for the context C_2 it is

$$\Lambda_2 = \Lambda \times \Lambda_{\theta_1} \times \Lambda_{\theta_3},$$

for the context C_3 it is

$$\Lambda_3 = \Lambda \times \Lambda_{\theta_2} \times \Lambda_{\theta_3}.$$

And, of course, we should consider three probability measures

$$dP_1(\lambda, \lambda_{\theta_1}, \lambda_{\theta_2}), dP_2(\lambda, \lambda_{\theta_1}, \lambda_{\theta_3}), dP_3(\lambda, \lambda_{\theta_2}, \lambda_{\theta_3}).$$

Random variables are functions on corresponding spaces

$$a_{\theta_1}(\lambda, \lambda_{\theta_1}), a_{\theta_2}(\lambda, \lambda_{\theta_2}), a_{\theta_3}(\lambda, \lambda_{\theta_3}).$$

Of course, Bell’s “condition of locality” is satisfied (otherwise we would have e.g. $a_{\theta_1}(\lambda, \lambda_{\theta_1}, \lambda_{\theta_2}), a_{\theta_2}(\lambda, \lambda_{\theta_2}, \lambda_{\theta_1})$ for the context C_1).

In this situation one should have strong arguments to assume that these three probability distributions could be obtained from a single probability measure

$$dP_1(\lambda, \lambda_{\theta_1}, \lambda_{\theta_2}, \lambda_{\theta_3})$$

on the space

$$\Lambda = \Lambda \times \Lambda_{\theta_1} \times \Lambda_{\theta_2} \times \Lambda_{\theta_3}.$$

5 Consequences for quantum mechanics

Finally, we come to quantum mechanics. Our contextual analysis of the EPR-Bohm experiment implies that the most natural explanation of nonexistence of a single probability space is that *the wave function does not determine probability in quantum mechanics* (in contrast to Bell’s assumption). We recall that Born’s rule contains not only the ψ -function but also spectral families of commutative operators which are measured simultaneously. Hence, the probability distribution is determined by the ψ -function as well as spectral families, i.e., observables.

Such an interpretation of mathematical symbols of the quantum formalism does not imply neither nonlocality nor “death of reality.”⁶

6 Bell’s Inequality and Negative and P-adic Probabilities

By looking for a trace in physics of the Boole-Vorobjev conclusion on nonexistence of probability one can find that this problem was intensively discussed, but in rather unusual form (at least from the mathematical viewpoint). During our conversations on the probabilistic structure of Bell’s inequality Alain Aspect permanently pointed out to a probabilistic possibility to escape Bell’s alternative: either local realism or quantum mechanics. This possibility mentioned by Alain Aspect is consideration of negative valued probabilities. A complete review on solving “Bell’s paradox” with the aid of negative probabilities was done by Muckenheim [68]. Although negative probabilities are meaningless from the mathematical viewpoint (however, see [69]- [76] for an attempt to define them mathematically by using p -adic analysis), there is some point in consideration of negative probabilities by physicists. In the light of our previous studies this activity can be interpreted as a sign of understanding that “normal probability distribution” does not exist. Surprisingly, but negative probability approach to Bell’s inequality can be considered as a link to Boole-Vorobjev’s viewpoint on violation of Bell’s inequality. Of course, the formal mathematical description by using negative probabilities does not have any reasonable physical interpretation

7 Detectors Efficiency

Another trace of nonexistence of a single probability space can be found in physical literature on detectors efficiency [86]– [89]. Theoreticians as well as experimenters are well aware about the fact that the real experiments induce huge losses of photons. Even if one associates (as Bell did) one fixed probability distribution with the source (the initial state), there are no reasons to

⁶One should not accuse the author in critique of J. Bell. J. Bell by himself did a similar thing with the von Neumann no-go theorem [], see [], by pointing out that some assumptions of von Neumann were unphysical.

assume that it is preserved by detectors.⁷

How reasonable is this attempt to explain violation of Bell's inequality by inefficiency of detectors?

I discussed this problem with many outstanding physicists. The common viewpoint was expressed in the reply of Alain Aspect to my question. He did not believe that interference-like behavior of correlations in the two slit experiment is just a consequence of inefficiency of detectors. For him (as well as for majority of physicists) violation of Bell's inequality is a consequence of fundamental quantumness of the EPR-Bohm experiment and not at all a technological problem of detectors efficiency. Similar viewpoint was presented by Philip Pearle, the first who paid attention to the possibility to simulate the EPR-Bohm correlations via detectors inefficiency [86].

Now we would like to explore another viewpoint to the EPR-Bohm correlations. If one accept that the correlations given by the EPR-Bohm experiment are nothing else than a special exhibition of the general interference phenomenon, then it would be surprising that for photons interference is generated by inefficiency of detectors, but for e.g. electrons it is a consequence of another hidden mechanism (we proceed our discussion under the assumption that quantum mechanics is not complete).⁸

Therefore, although improvement of detectors efficiency is very important for quantum foundations [91], [92] one could not expect that the EPR-Bohm-Bell paradox would be resolved via approaching approximately 100% detectors efficiency.

⁷We recall that our basic hypothesis is that Bell's inequality is violated due to non-Kolmogorovness. We discuss different sources of nonexistence of a single probability space. One of such possible sources is inefficiency of detectors.

⁸We remark that rather common viewpoint is that the EPR-Bohm experiment in the Bell's framework is essentially new experiment comparing with the two slit interference experiment. It is often mentioned “old quantum mechanics” before Bell and “new quantum mechanics” coupled to violation of Bell's inequality. However, purely mathematically these correlations can be obtained as a special case of interference of probabilities, see [90]. The point of view that fundamentally the EPR-Bohm experiment in the Bell's framework and the two slit experiment demonstrate the same physical phenomenon – interference – was presented by a number of theoreticians and experimenters [18]– [23].

8 Fair sampling

The assumption of fair sampling is typically misidentified with detectors efficiency. However, they are essentially different. Unfair sampling could take place even for detectors having 100% efficiency.

By the fair sampling assumption ensembles of pairs $\omega = (\omega_1, \omega_2)$ of output photons from two polarization beam splitters have the same probabilistic properties independently on orientations of splitters. Thus one can operate with a single probability distribution.

However, a priory there are no reasons to assume that ensembles of pairs of photons which pass polarization beam splitters for different choices of orientations (and were identified as pairs by using time window, see e.g. [1] for details) have identical statistical properties, see e.g. [93],[94]. Unfair sampling implies that a single probability which would serve all orientations does not exist.

Of course, one should construct a physical model of unfair sampling process which might be performed by polarization beam splitters. Moreover, one should explain, cf. with remark on detectors efficiency, why the photon interference is a consequence of unfair sampling, but e.g. electron interference is a consequence of something else.

Finally, we remark that in general nonexistence of probability is not reduced to inefficiency of detectors or unfair sampling.

9 Extended semantic realism

This is a generalization of quantum formalism, see Garola and Solombrino [33]–[35], Sozzo [66], by which each quantum observable gets an additional point to its spectrum, say a_0 , denoting the event of nonregistration. In principle, one could consider extended semantic realism as a possible formalization of unfair sampling. However, this approach suffers from the same problem as the efficiency detectors viewpoint. One should find a physical model explaining nonobservations of non negligible subensembles of systems.

10 Anti-photon interpretation of violation of Bell's inequality

All real experiments demonstrating violation of Bell's inequality and at the same time guaranteeing locality have been done for photons. However, since first days of quantum mechanics, many prominent quantum physicists criticized Einstein's proposal of photon, e.g. Lande [97], [98] and Lamb [99]. For example, Lamb wrote in his "Anti-photon" [99] p. 221: "It is high time to give up the use of the word "photon", and of a bad concept which will shortly be a century old. Radiation does not consist of particles ..."

The idea about purely wave structure of classical as well as quantum light has interesting consequences in the EPR-Bohm experimental framework. If one rejects the conventional picture of detectors registering particles (photons) and if one considers detectors as devices integrating continuously (up to a certain threshold) electromagnetic radiation, then the whole Bell's representation loses its meaning. We could not associate hidden variables to clicks of detectors. We could not consider an ensemble of photons produced by a source and the corresponding probability distribution. This idea was explored in different versions by Santos [100],[101] , Thompson, Kracklauer [102], [103] and Roychoudhuri.

It is typically used the semiclassical model: light is classical, but atoms are quantum. In this model the electromagnetic field is not quantized by itself, but exchange of energy is performed by discrete portions – quanta. It is not easy to reject completely the idea that violation of Bell's inequality implies simply that one should use the semiclassical model, instead of the completely quantum one. All detectors are based on either scattering of electrons by photons (photomultipliers tubes – PMTs) from a photodiode or creation by photons pairs electron-hole (avalanche photodiodes – APDs and the visible light counters – VLPs), see [91] for a detailed review. Thus photon-like discreteness of counting might be just an illusion induced by discreteness of electron emission. The latter one might be explained by the semiclassical model (as well as e.g. the photoelectric effect).

If detectors interact with the continuous electromagnetic field then the picture statistical ensembles of photons which was used by Bell is misleading. For example, the signal field could produce photon-like counts via combination with noise and even vacuum fluctuations.

New possibilities to test the anti-photon interpretation of violation of

Bell's inequality is provided by Tungsten-based Superconducting Transition-Edge Sensors (W-TESs), see [92]. These are ultra-sensitive microcalorimeters. It seems that such detectors absorb even portions of photons (if the latter exist). In contrast to PMTs, APDs and VLPCs, W-TESs functioning is not based on the threshold principle. It seems that W-TESs provide access directly to energy of prequantum classical electromagnetic field (of course, if such nonquantized field really exists). PMTs, APDs and VLPCs react only to integral portions of energy $E_n = nh\nu$, where n is the number of photons in the pulse and ν is the frequency of light (in accordance with quantum theory $E = h\nu$ is photon energy). In the anti-photon framework a pulse can contain some random portions of photon, $E_n(\omega) = E_n + \xi(\omega)$, where $\xi(\omega)$ is a random variable (here ω is a random parameter) and $|\xi(\omega)| < h\nu$. In principle, the EPR-Bohm correlations can be reproduced as the result of cutoff of this random term. We emphasize that $\xi(\omega)$ is not detector's noise. This is a part of the original signal. Of course, $\xi(\omega)$ can interact with noises. If detector's noise is not so high, then one can hope to extract "nonquantum part" of the signal. If a special ("no classical character") of the EPR-Bohm correlations was really a consequence of detection cutoff, then by taking into account this term we might expect to reproduce classical correlations which would not violate Bell's inequality.

The main problem of the anti-photon interpretation of violation of Bell's inequality is impossibility to generalize this argument to massive particles, cf. inefficiency of detectors and unfair sampling. Adherents of this interpretation, e.g. Santos [100], [101], typically point out that local experiments with massive particles violating Bell's inequality have never been done. This is an important argument. The majority of experimenters see the locality loophole in the famous Boulder experiment [104].

However, if one accepts the viewpoint that the EPR-Bohm experiment is a special case of interference experiments, then he should also explain why the Copenhagen postulate on wave-particle duality could not be applied to light, but it should be applied to massive particles – to explain interference of such particles.

We remark that Alfred Lande [97], [98] presented detailed description of the interference effects for massive particles without using the wave-particle duality. For him massive particles are just particles, but electromagnetic field is just field. If one generalizes Lande's argument to the EPR-Bohm experiment, he should be able to obtain the EPR-Bohm correlations for the

electron spin by using the purely particle picture.

Thus the only possibility to interpret the EPR-Bohm experiment by rejecting Bohr's principle of complementarity (and hence the Copenhagen interpretation) is to create a purely wave model of the EPR-Bohm correlations for experiments with light and a purely particle model for experiments with massive particles.

11 Anomalies in experimental data

We found [93], [94] that the experimental correlations for photon polarization have an intriguing property. In the experimental data there are visible non-negligible deviations of “experimental probabilities” (frequencies):

$$P_{++}^{\text{exp}}(\theta_1, \theta_2), P_{+-}^{\text{exp}}(\theta_1, \theta_2), P_{-+}^{\text{exp}}(\theta_1, \theta_2), P_{--}^{\text{exp}}(\theta_1, \theta_2)$$

from the predictions of quantum mechanics, namely,

$$P_{++}(\theta_1, \theta_2) = P_{--}(\theta_1, \theta_2) = 1/2 \cos^2(\theta_1 - \theta_2)$$

and

$$P_{+-}(\theta_1, \theta_2) = P_{-+}(\theta_1, \theta_2) = 1/2 \sin^2(\theta_1 - \theta_2).$$

However, in some mysterious way those deviations compensate each other and finally the correlation

$$E^{\text{exp}}(\theta_1, \theta_2) = P_{++}^{\text{exp}}(\theta_1, \theta_2) - P_{+-}^{\text{exp}}(\theta_1, \theta_2) - P_{-+}^{\text{exp}}(\theta_1, \theta_2) + P_{--}^{\text{exp}}(\theta_1, \theta_2)$$

is in the complete agreement with the QM-prediction, namely,

$$E(\theta_1, \theta_2) = P_{++}(\theta_1, \theta_2) - P_{+-}(\theta_1, \theta_2) - P_{-+}(\theta_1, \theta_2) + P_{--}(\theta_1, \theta_2) = \cos 2(\theta_1 - \theta_2).$$

Therefore such anomalies play no role in the Bell's inequality framework. Nevertheless, other linear combinations of experimental probabilities do not have such a compensation property. There can be found non-negligible deviations from predictions of quantum mechanics. Thus neither classical nor quantum model can pass the whole family of statistical tests given by all possible linear combinations of the EPR-Bohm probabilities.

Does it mean that both models are wrong?

12 Eberhard-Bell Theorem

In quantum information community rather common opinion is that one could completely exclude probability distributions from derivation of Bell's inequality and proceed by operating with frequencies. One typically refers to the result of works [12]–[14] which we shall call the Eberhard-Bell theorem (in fact, the first frequency derivation of Bell's inequality was done by Stapp[11], thus it may be better to speak about Bell-Stapp-Eberhard theorem). By this theorem Bell's inequality can be obtain only under assumptions of realism – the maps $\lambda \rightarrow a_\theta(\lambda)$ is well defined – and locality – the random variable $a_\theta(\lambda)$ does not depend on other variables which are measured simultaneously with it. Thus (in opposite to the original Bell derivation) existence of the probability measure P serving for all polarization (or spin) projections is not assumed.

At the first sight it seems that our previous considerations have no relation to the Eberhard-Bell theorem. One might say: “Yes, Bell proceeded wrongly, but his arguments are still true, because they were justified by Eberhard in the frequency framework.”

As was shown [26], the use of frequencies, instead of probabilities, does not improve Bell's considerations, see also Hess and Philipp [36]. The contextual structure of the EPR-Bohm experiment plays again the crucial role. If we go into details of Eberhard's proof, we shall immediately see that he operated with statistical data obtained from three different experimental contexts, C_1, C_2, C_3 , in such a way as it was obtained on the basis of a single context. He took results belonging to one experimental setup and add or subtract them from results belonging to another experimental setup. These are not proper manipulations from the viewpoint of statistics. One never performs algebraic mixing of data obtained for totally different sample. Thus if one wants to proceed in Eberhard's framework, he should find some strong reasons that the situation in the EPR-Bohm experiment differs crucially from the general situation in statistical experiments. I do not see such reasons. Moreover, the EPR-Bohm experimental setup is very common from the general statistical viewpoint.

Moreover, Eberhard's framework pointed to an additional source of nonexistence of a single probability distribution, see De Baere [77] and also [78]–[80]. Even if we ignore the contribution of measurement devices, then the ψ -function still need not determine a single probability distribution. In Eberhard's framework we should operate with results which are obtained in dif-

ferent runs. One could ask: Is it possible to guarantee that different runs of experiment produce the same probability distribution of hidden parameters? It seems that there are no reasons for such an assumption. We are not able to control the source on the level of hidden variables. It may be that the ψ -function is just a symbolic representation of the source, but it represents a huge ensemble of probability distributions of hidden variables. If e.g. hidden variables are given by classical fields, see e.g. [81]–[83], then a finite run of realizations (emissions of entangled photons) may be, but may be not representative for the ensemble of hidden variables produced by the source.

13 Comparing of the EPR and the EPR-Bohm experiments

Typically the original EPR experiment [84] for correlations of coordinates and momenta and the EPR-Bohm experiment for spin (or polarization) projections are not sharply distinguished. People are almost sure that it is the same story, but the experimental setup was modified to move from “gedanken experiment” to real physical experiment. However, it was not the case! We should sharply distinguish these two experimental frameworks.

The crucial difference between the original EPR experiment and a new experiment which was proposed by Bohm is that these experiments are based on quantum states having essentially different properties. The original EPR state

$$\Psi(x_1, x_2) = \int_{-\infty}^{\infty} \exp \left\{ \frac{i}{\hbar} (x_1 - x_2 + x_0)p \right\} dp,$$

and the singlet state

$$\psi = \frac{1}{\sqrt{2}} (|+> |-> - |-> |+>)$$

which is used in the EPR-Bohm experiment have in common only one thing: they describe correlated (or by using the modern terminology entangled) systems. But, in contrast to the EPR-Bohm state, one can really (as EPR claimed) associate with the original EPR state a single probability measure describing incompatible quantum observables (position and momentum). The rigorous prove in probabilistic terms was proposed by the author

and Igor Volovich in [85]. On the other hand, as we have seen for the singlet state one could not construct a probabilistic model describing elements of reality corresponding to incompatible observables.

Thus the original EPR state is really exceptional from the general viewpoint of statistical analysis. But the EPR-Bohm state behaves “normally.” In fact, there is no clear physical explanation why statistical data for incompatible contexts can be based on a single Kolmogorov space in one case and not in another. One possible explanation is that “nice probabilistic features of the original EPR-experiment” arise only due to the fact that it is “gedanken experiment.”

14 Legget’s inequality and tests for nonlocal realistic theories

Here we follow Legget [95] and Gröblacher et al.[96], so details can be found in cited papers. The source is assumed to distribute pairs of well polarized photons. The size of the sub-ensemble in which photons have polarizations u and v , respectively, is described by the density $F(u, v)$. Individual measurement outcomes are determined by a hidden variable λ (which may have a huge dimension; in fact, it may belong to an infinite-dimensional space – e.g. for a classical field type hidden variables, see [105]). For the fixed polarizations u and v , the density of hidden variables is given by $\rho_{u,v}(\lambda)$.

The dichotomous (\pm) measurement results are given by random variables $A(a, b, u, v, \lambda)$ and $B(a, b, u, v, \lambda)$, where a and b are settings of polarization beam splitters of Alice and Bob, respectively. The main Legget’s trick (which was repeated by Gröblacher et al.) is that the average of AB is calculated into the two steps:

- 1). The average with respect to $\rho_{u,v}(\lambda)$.
- 2). The average with respect to $F(u, v)$.

By using after the first step some algebraic manipulations and then averaging according to the second step, Legget obtained a Bell-type inequality. Similar inequality was considered in [96] and tested experimentally.

The main problem of this derivation is that, instead of the rigorous mathematical operation with conditional densities, we see formal manipulation with densities $F(u, v)$ and $\rho_{u,v}(\lambda)$. What is the real meaning of $\rho_{u,v}(\lambda)$ in

the rigorous mathematical framework? This is nothing else than the *conditional density* $\rho(\lambda|u, v)$. If one takes this fact into account, it would be immediately clear that Legget's derivation suffers of the same problem as the Bell's original one. It could be possible only under the assumption of probabilistic compatibility of random variables $A(a, b, u, v, \lambda), B(a, b, u, v, \lambda)$ for all settings a, b involved in considerations.

To simplify presentation, let us consider discrete hidden variable λ and some discrete sampling of polarizations u and v (the latter is consistent with [96]). Thus Legget's considerations have the form. Set:

$$\overline{AB}(u, v) \equiv \sum_{\lambda} A(a, b, u, v, \lambda), B(a, b, u, v, \lambda) \rho_{u,v}(\lambda) \quad (1)$$

and

$$\langle AB \rangle = \sum_{u,v} \overline{AB}(u, v) F(u, v). \quad (2)$$

Thus

$$\langle AB \rangle = \sum_{u,v,\lambda} A(a, b, u, v, \lambda), B(a, b, u, v, \lambda) \rho_{u,v}(\lambda) F(u, v). \quad (3)$$

The crucial point of Legget's considerations is that he assumes that the latter expression coincides with the classical probabilistic average $E(AB)$ of the product AB . However, the latter is valid only under assumption that there exists a probability distribution $P(u, v, \lambda)$ such that

$$F(u, v) = \sum_{\lambda} P(u, v, \lambda),$$

is the marginal probability and

$$\rho_{u,v}(\lambda) \equiv \rho(\lambda|u, v) = \frac{P(u, v, \lambda)}{F(u, v)}$$

is the conditional probability. Under such assumptions

$$\begin{aligned} \langle AB \rangle &= \sum_{u,v,\lambda} A(a, b, u, v, \lambda), B(a, b, u, v, \lambda) \rho_{u,v}(\lambda) F(u, v) \\ &= E(AB) = \sum_{u,v,\lambda} A(a, b, u, v, \lambda), B(a, b, u, v, \lambda) P(u, v, \lambda). \end{aligned} \quad (4)$$

Thus Legget's derivation is based on the (implicit) assumption: existence of the probability distribution $P(u, v, \lambda)$. Moreover, to proceed further to his inequality Legget (as well as Gröblacher et al.[96]) should assume that $P(u, v, \lambda)$ does not depend on settings a and b . Thus they again assume the probabilistic compatibility of random variables $A(a, b, u, v, \lambda), B(a, b, u, v, \lambda)$ for a family of settings a and b . We again do not see any physical or statistical reason for such an assumption.

15 Appendix: Proofs

15.1 Bell's inequality

Let $\mathcal{P} = (\Lambda, \mathcal{F}, P)$ be a Kolmogorov probability space: Λ is the set of parameters, \mathcal{F} is a σ -algebra of its subsets (used to define a probability measure), P is a probability measure. For any pair of random variables $u(\lambda), v(\lambda)$, their covariation is defined by

$$\langle u, v \rangle = \text{cov}(u, v) = \int_{\Lambda} u(\lambda) v(\lambda) d\mathbf{P}(\lambda).$$

We reproduce the proof of Bell's inequality in the measure-theoretic framework.

Theorem. (Bell inequality for covariations) *Let $a, b, c = \pm 1$ be random variables on \mathcal{P} . Then Bell's inequality*

$$|\langle a, b \rangle - \langle c, b \rangle| \leq 1 - \langle a, c \rangle \quad (5)$$

holds.

Proof. Set $\Delta = \langle a, b \rangle - \langle c, b \rangle$. By linearity of Lebesgue integral we obtain

$$\begin{aligned} \Delta &= \int_{\Lambda} a(\lambda)b(\lambda)d\mathbf{P}(\lambda) - \int_{\Lambda} c(\lambda)b(\lambda)d\mathbf{P}(\lambda) \\ &= \int_{\Lambda} [a(\lambda) - c(\lambda)]b(\lambda)d\mathbf{P}(\lambda). \end{aligned} \quad (6)$$

As

$$a(\lambda)^2 = 1, \quad (7)$$

we have:

$$|\Delta| = \left| \int_{\Lambda} [1 - a(\lambda)c(\lambda)]a(\lambda)b(\lambda)d\mathbf{P}(\lambda) \right| \quad (8)$$

$$\leq \int_{\Lambda} [1 - a(\lambda)c(\lambda)]dP(\lambda).$$

It is evident that “hidden Bell’s postulate” on the existence of a single probability measure P serving for three different experimental contexts (probabilistic compatibility of three random variables) plays the crucial role in derivation of Bell’s inequality.

15.2 Wigner inequality

We recall the following simple mathematical result, see Wigner [7]:

Theorem 1.2. (Wigner inequality) *Let $a, b, c = \pm 1$ be arbitrary random variables on a Kolmogorov space \mathcal{P} . Then the following inequality holds:*

$$\begin{aligned} P(a = +1, b = +1) + P(b = -1, c = +1) \\ \geq \mathbf{P}(a = +1, c = +1). \end{aligned} \quad (9)$$

Proof. We have:

$$\begin{aligned} P(a(\lambda) = +1, b(\lambda) = +1) \\ = P(a(\lambda) = +1, b(\lambda) = +1, c(\lambda) = +1) \\ + P(a(\lambda) = +1, b(\lambda) = +1, c(\lambda) = -1), \end{aligned} \quad (10)$$

$$\begin{aligned} P(b(\lambda) = -1, c(\lambda) = +1) \\ = P(a(\lambda) = +1, b(\lambda) = -1, c(\lambda) = +1) \\ + P(\lambda \in \Lambda : a(\lambda) = -1, b(\lambda) = -1, c(\lambda) = +1), \end{aligned} \quad (11)$$

and

$$\begin{aligned} P(a(\lambda) = +1, c(\lambda) = +1) \\ = P(a(\lambda) = +1, b(\lambda) = +1, c(\lambda) = +1) \\ + P(a(\lambda) = +1, b(\lambda) = -1, c(\lambda) = +1). \end{aligned} \quad (12)$$

If we add together the equations (10) and (11) we obtain

$$\begin{aligned}
& P(a(\lambda) = +1, b(\lambda) = +1) + P(b(\lambda) = -1, c(\lambda) = +1) \\
&= P(a(\lambda) = +1, b(\lambda) = +1, c(\lambda) = +1) \\
&\quad + P(a(\lambda) = +1, b(\lambda) = +1, c(\lambda) = -1) \tag{13} \\
&\quad + P(a(\lambda) = +1, b(\lambda) = -1, c(\lambda) = +1) \\
&\quad + P(a(\lambda) = -1, b(\lambda) = -1, c(\lambda) = +1).
\end{aligned}$$

But the first and the third terms on the right hand side of this equation are just those which when added together make up the term $P(a(\lambda) = +1, c(\lambda) = +1)$ (Kolmogorov probability is additive). It therefore follows that:

$$\begin{aligned}
& P(a(\lambda) = +1, b(\lambda) = +1) + \mathbf{P}(b(\lambda) = -1, c(\lambda) = +1) \\
&= P(a(\lambda) = +1, c(\lambda) = +1) \\
&\quad + P(a(\lambda) = +1, b(\lambda) = +1, c(\lambda) = -1) \tag{14} \\
&\quad + P(a(\lambda) = -1, b(\lambda) = -1, c(\lambda) = +1)
\end{aligned}$$

By using non negativity of probability we obtain the inequality:

$$\begin{aligned}
& P(a(\lambda) = +1, b(\lambda) = +1) + \mathbf{P}(b(\lambda) = -1, c(\lambda) = +1) \\
&\geq P(a(\lambda) = +1, c(\lambda) = +1) \tag{15}
\end{aligned}$$

It is evident that “hidden Bell’s postulate” on the existence of a single probability measure P serving for three different experimental contexts (probabilistic compatibility of three random variables) plays the crucial role in derivation of Wigner’s inequality.

16 Conclusion

In probability theory Bell’s type inequalities were studied during last hundred years as constraints for probabilistic compatibility of families of random

variables – possibility to realize them on a single probability space. In opposite to quantum physics, such arguments as nonlocality and “death of reality” were not involved in considerations. In particular, nonexistence of a single probability space does not imply that the realistic description (a map $\lambda \rightarrow a(\lambda)$) is impossible to construct. Bell’s type inequalities were considered as signs (sufficient conditions) of impossibility to perform simultaneous measurement *all random variables* from a family under consideration. Such an interpretation can be used for statistical data obtained in the EPR-Bohm experiment for entangled photons.

In any event, Bell’s inequality could not be used to obtain the definite conclusion on the relation between quantum and classical models.

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